

Skew-orthogonal polynomials: the quartic case.

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We present an iterative technique to obtain skew-orthogonal polynomials with quartic weight, arising in the study of symplectic ensembles of random matrices.

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1. INTRODUCTION

After the publication of [1] and [2], late Prof. M. L. Mehta was not particularly happy with the formal nature of the presentation. Following his suggestion, this paper attempts to give an explicit workout of the skew-orthogonal polynomials (SOP) corresponding to the quartic potential.

We study SOP arising in the study of symplectic ensembles of random matrices. The corresponding weight function is $w(x) = \exp[-V(x)]$, where

$$V(x) = \frac{x^4}{4} + \frac{\alpha x^2}{2}, \quad \alpha \in \mathbb{R}, \quad (1.1)$$

is the quartic potential. In this paper, we outline an iterative technique to develop these polynomials which can be used to obtain the level-density and 2-point function for the symplectic ensembles of random matrices. Here, we emphasize that this method can be easily extended to all potentials of the form

$$V(x) = \sum_{k=1}^d \frac{u_{2k} x^{2k}}{2k}, \quad u_{2d} = 1. \quad (1.2)$$

Without loss of generality, we will be dealing with monic SOP of the form

$$\phi_n(x) = w(x) \sum_{k=0}^n c_k^{(n)} x^k, \quad c_n^{(n)} = 1, \quad (1.3)$$

and define

$$\psi_n(x) := \frac{d}{dx} \phi_n(x). \quad (1.4)$$

They satisfy the skew-orthonormalization relation

$$\int_{\mathbb{R}} \phi_n(x) \psi_m(x) dx = g_n Z_{nm}, \quad (1.5)$$

where

$$Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dot{+} \dots \dot{+}, \quad (1.6)$$

is an anti-symmetric block-diagonal matrix with $Z^2 = -1$. g_n is the normalization constant with the property

$$g_{2n} = g_{2n+1}. \quad (1.7)$$

Here $\psi_n(x)$ is a polynomial of order $n + 2d - 1$ with leading coefficient -1 . We have dropped the superscript β (as in [1]) and [2]) since we will only be interested in SOP arising in the study of symplectic ensembles of random matrices.

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As explained in Ref.[1] and [2], we expand $\psi(x)$ in terms of $\phi(x)$ and write

$$x\psi_n(x) = \sum_{m=j}^k R_{nm}\phi_m(x), \quad (1.8)$$

where from Eqs.(1.2, 1.3, 1.4), one can say that $k = n + 2d$. Furthermore, using the integral $\int_{\mathbb{R}} x\psi_n(x)\psi_m(x)dx$, we can show that the matrix R satisfy anti-self dual relation

$$R = ZR^tZ \equiv -R^D. \quad (1.9)$$

Summing up these results, we can write for any polynomial weight

$$x\psi_{2n}(x) = R_{2n,2n+2d}\phi_{2n+2d}(x) + \dots + R_{2n,2n-2d}\phi_{2n-2d}(x), \quad (1.10)$$

$$x\psi_{2n+1}(x) = R_{2n+1,2n+2d+1}\phi_{2n+2d+1}(x) + \dots + R_{2n+1,2n-2d}\phi_{2n-2d}(x), \quad (1.11)$$

where $\phi_{-m} = 0$, m being a positive integer. Our choice of the potential ($u_{2d} = 1$) ensures that

$$R_{m,m+2d} = -1, \quad m \geq 0. \quad (1.12)$$

Also from Eq.(1.9), we can show that

$$\begin{aligned} R_{2n,2n-2d} = -R_{2n-2d+1,2n+1} &= 1, & n \geq d, \\ &= 0, & n < d, \\ R_{2n+1,2n-2d+1} = -R_{2n-2d,2n} &= 1, & n \geq d, \\ &= 0, & n < d. \end{aligned} \quad (1.13)$$

Following this brief recapitulation of [1] and [2], we will outline our plan of action.

1. We will calculate the SOP's $\phi_n(x)$ for $0 \leq n < 2d$ (for the particular case $d = 2$) using generalized Gram-Schmidt orthogonalization.
2. Using these polynomials, we will use Eqs.(1.10) and (1.11) to calculate the higher order polynomials recursively. The coefficients will be expressed in terms of integrals of the form $\int_{\mathbb{R}} x^n \exp[-2V(x)]dx$.
3. We show that these coefficients themselves satisfy a set of difference equations.
4. Using these difference equations, we also obtain the normalization constants.
5. Finally, we talk briefly about the zeros of these polynomials.

2. THE GRAM-SCHMIDT TECHNIQUE

Putting $n = 0$ in Eqs.(1.10) and (1.11) (keeping in mind Eq.(1.13)), we can see that it is useless, unless we have information about the SOP's ϕ_m for $2d > m \geq 0$. To overcome this problem, we will obtain these $2d$ polynomials ($\phi_0(x), \dots, \phi_{2d-1}(x)$) using the Gram-Schmidt technique for SOP. From here onward, we will focus our attention on the specific weight function defined in Eq.(1.1), although the method outlined can be extended to any d . For $d = 2$, the first 4 monic polynomials can be written as

$$\phi_0(x) = w(x), \quad \phi_1(x) = xw(x), \quad (2.1)$$

$$\phi_2(x) = (x^2 + c_0^{(2)})w(x), \quad \phi_3(x) = (x^3 + c_1^{(3)})w(x). \quad (2.2)$$

Correspondingly

$$\psi_0(x) = -V'(x)w(x), \quad \psi_1(x) = (1 - xV'(x))w(x), \quad (2.3)$$

$$\psi_2(x) = [2x - (x^2 + c_0^{(2)})V'(x)]w(x), \quad \psi_3(x) = [3x^2 + c_1^{(3)} - V'(x)(x^3 + c_1^{(3)}x)]w(x). \quad (2.4)$$

Using

$$\int_{-\infty}^{\infty} \phi_2(x)\psi_1(x)dx = 0, \quad (2.5)$$

we get

$$C_0^{(2)} = -\frac{\int_{-\infty}^{\infty} x^2(1 - xV'(x))w^2(x)dx}{\int_{-\infty}^{\infty} (1 - xV'(x))w^2(x)dx}. \quad (2.6)$$

Similarly, using

$$\int_{-\infty}^{\infty} \phi_3(x)\psi_0(x)dx = 0, \quad (2.7)$$

we get

$$C_1^{(3)} = -\frac{\int_{-\infty}^{\infty} x^3V'(x)w^2(x)dx}{\int_{-\infty}^{\infty} xV'(x)w^2(x)dx}. \quad (2.8)$$

We also get

$$g_0 = g_1 = \int_{-\infty}^{\infty} \phi_0(x)\psi_1(x)dx, \quad g_2 = g_3 = \int_{-\infty}^{\infty} \phi_2(x)\psi_3(x)dx. \quad (2.9)$$

3. RECURSION RELATION

Now, let us look at Eqs.(1.10) and (1.11). Since $V(x) = V(-x)$, we have

$$\psi_{2n}(x) = -\psi_{2n}(-x), \quad \psi_{2n+1}(x) = \psi_{2n+1}(-x). \quad (3.1)$$

Thus the odd (even) terms will be absent in Eqs.(1.10) ((1.11)) since

$$R_{2n,2n+2k+1} = -\frac{1}{g_{2n+2k}} \int_{-\infty}^{\infty} x\psi_{2n}(x)\psi_{2n+2k}(x)dx = 0,$$

and

$$R_{2n+1,2n+2k} = \frac{1}{g_{2n+2k}} \int_{-\infty}^{\infty} x\psi_{2n+1}(x)\psi_{2n+2k+1}(x)dx = 0. \quad (3.2)$$

Using these results and Eq.(1.12), we can rewrite Eqs. (1.10) and (1.11) for $n = 0$ and $d = 2$ as

$$x\psi_0(x) = -\phi_4(x) + R_{0,2}\phi_2(x) + R_{0,0}\phi_0(x), \quad (3.3)$$

$$x\psi_1(x) = -\phi_5(x) + R_{1,3}\phi_3(x) + R_{1,1}\phi_1(x). \quad (3.4)$$

Also, using the skew-orthogonality property, we get

$$R_{0,0} = \frac{1}{g_0} \int_{-\infty}^{\infty} x\psi_0(x)\psi_1(x)dx, \quad (3.5)$$

$$R_{0,2} = \frac{1}{g_2} \int_{-\infty}^{\infty} x\psi_0(x)\psi_3(x)dx. \quad (3.6)$$

Similarly,

$$R_{1,1} = -\frac{1}{g_0} \int_{-\infty}^{\infty} x\psi_0(x)\psi_1(x)dx = -R_{0,0}, \quad (3.7)$$

$$R_{1,3} = -\frac{1}{g_2} \int_{-\infty}^{\infty} x\psi_1(x)\psi_2(x)dx. \quad (3.8)$$

Here, we note that $R_{0,0} = -R_{1,1}$ can be obtained directly from Eq.(1.9). Once the coefficients are known, one can calculate $\phi_4(x)$ and $\phi_5(x)$ using Eqs. (3.3) and (3.4). Using (1.4), we can calculate $\psi_4(x)$ and $\psi_5(x)$. We can also calculate

$$g_4 = g_5 = \int_{-\infty}^{\infty} \phi_4(x)\psi_5(x)dx. \quad (3.9)$$

We will now calculate $\phi_6(x)$ and $\phi_7(x)$ using

$$x\psi_2(x) = -\phi_6(x) + R_{2,4}\phi_4(x) + R_{2,2}\phi_2(x) + R_{2,0}\phi_0(x), \quad (3.10)$$

$$x\psi_3(x) = -\phi_7(x) + R_{3,5}\phi_5(x) + R_{3,3}\phi_3(x) + R_{3,1}\phi_1(x). \quad (3.11)$$

Again, using Eq.(1.9), we have

$$R_{2,0} = -R_{1,3}, \quad R_{3,1} = -R_{0,2}, \quad R_{2,2} = -R_{3,3}. \quad (3.12)$$

$R_{0,2}$ and $R_{1,3}$ has already been calculated in (3.6) and (3.8) respectively. Also

$$R_{2,4} = \frac{1}{g_4} \int_{-\infty}^{\infty} x\psi_2(x)\psi_5(x)dx, \quad (3.13)$$

$$R_{3,5} = -\frac{1}{g_4} \int_{-\infty}^{\infty} x\psi_3(x)\psi_4(x)dx, \quad (3.14)$$

and

$$R_{2,2} = -R_{3,3} = \frac{1}{g_2} \int_{-\infty}^{\infty} x\psi_2(x)\psi_3(x)dx. \quad (3.15)$$

With these, we can calculate $\phi_6(x)$, $\phi_7(x)$ and correspondingly $\psi_6(x)$ and $\psi_7(x)$ and have

$$g_6 = g_7 = \int_{-\infty}^{\infty} \phi_6(x)\psi_7(x)dx. \quad (3.16)$$

Following the same technique, we can obtain the polynomials for all n . For $n \geq 2$ and $d = 2$, we may rewrite Eqs.(1.10) and (1.11) as

$$x\psi_{2n}(x) = \sum_{m=-2}^2 R_{2n,2n+2m}\phi_{2n+2m}(x), \quad x\psi_{2n+1}(x) = \sum_{m=-2}^2 R_{2n+1,2n+2m+1}\phi_{2n+2m+1}(x). \quad (3.17)$$

From Eq.(1.12), we have

$$R_{2n,2n+4} = R_{2n+1,2n+5} = -1. \quad (3.18)$$

Also from Eq.(1.13), we have

$$R_{2n,2n-4} = -R_{2n-3,2n+1} = 1, \quad R_{2n+1,2n-3} = -R_{2n-4,2n} = 1. \quad (3.19)$$

$R_{2n,2n-2}$ (for the even case) and $R_{2n+1,2n-1}$ (for the odd case) is already known since

$$R_{2n,2n-2} = -R_{2n-1,2n+1}, \quad R_{2n+1,2n-1} = -R_{2n-2,2n}. \quad (3.20)$$

Finally, we are left with terms of the form $R_{k,k+2}$ and $R_{k,k}$ (for both k odd and even). They are given by

$$R_{2n,2n+2k} = \frac{1}{g_{2k+2n}} \int_{-\infty}^{\infty} x\psi_{2n}(x)\psi_{2n+2k+1}(x)dx, \quad k = 0, 1 \quad (3.21)$$

$$R_{2n+1,2n+2k+1} = -\frac{1}{g_{2k+2n}} \int_{-\infty}^{\infty} x\psi_{2n+1}(x)\psi_{2n+2k}(x)dx, \quad k = 0, 1. \quad (3.22)$$

Here, we note that by expanding $x\psi_{2n}$ (or $x\psi_{2n+1}$), we can evaluate $R_{2n,2n+2k}$ (or $R_{2n+1,2n+2k+1}$) analytically. But that will involve terms like $c_{2n+2}^{(2n+4)}$ (or $c_{2n+3}^{(2n+5)}$) and thereby cannot be used to obtain the polynomials recursively. We might recall that for monic orthogonal polynomials, this problem does not exist [5]. However it is possible to evaluate Eqs.(3.21) and (3.22) numerically, although it may be a tiresome process.

4. THE NORMALIZATION CONSTANT

So far, we have outlined a formalism to obtain the polynomials recursively, where the recursion coefficients are expressed in terms of certain integrals which needs to be evaluated at every iteration. It is not practical to use this process for the study of large n behavior of these polynomials. Nor is it convenient to study potentials with larger d , since the number of terms in the recursion relation increases with d .

In this section, we present an alternative technique to obtain both the recursion coefficients and the normalization constant. We begin with the identity (obtained by integration by parts)

$$\int [x(x\psi_j(x))']' \phi_k(x) dx = - \int (x\psi_k(x))(x\psi_j(x))' dx, \quad j, k = 0, 1, \dots \quad (4.1)$$

The only criteria to use this formalism is that as initial condition, we need to know these polynomials for $n = 0, \dots, 2d-1$. Taking $j = 2n$ and $k = 2n+1$ in Eq.(4.1), we get the recursion relation for the normalization constant. We get

$$g_4 + \gamma_0 g_2 - (1 + \gamma_0) g_0 = 0, \quad n = 0, \quad (4.2)$$

$$g_6 + \gamma_1 g_4 - (1 + \gamma_0 + \gamma_1) g_2 + \gamma_0 g_0 = 0, \quad n = 1, \quad (4.3)$$

$$g_{2n+4} + \gamma_n g_{2n+2} - (2 + \gamma_n + \gamma_{n-1}) g_{2n} + \gamma_{n-1} g_{2n-2} + g_{2n-4} = 0, \quad n \geq 2, \quad (4.4)$$

where

$$\gamma_n := R_{2n, 2n+2} R_{2n+1, 2n+3}. \quad (4.5)$$

Here, we have expanded $x\psi_{2n}(x)$ and $x\psi_{2n+1}(x)$ (using Eq.(3.17)), and using Eqs.(3.18), (3.19) and (3.20)), get the result. We have assumed ϕ_{-n} and consequently g_{-n} and γ_{-n} is zero, n being a positive integer. Here, we must also remember that for these SOP, $g_{2n} = g_{2n+1}$.

To evaluate γ , we need to know $R_{k, k+2}$ for both k odd and even. This can be calculated from Eq.(4.1) by putting $j = 2n$, $k = 2n+3$ and $j = 2n+1$, $k = 2n+2$. We get

$$R_{2n+3, 2n+5}(g_{2n+4} - g_{2n+2}) - R_{2n+1, 2n+1} R_{2n, 2n+2}(g_{2n} - g_{2n+2}) + R_{2n-1, 2n+1}(g_{2n-2} - g_{2n+2}) = 0, \quad (4.6)$$

$$R_{2n+2, 2n+4}(g_{2n+4} - g_{2n+2}) - R_{2n, 2n} R_{2n+1, 2n+3}(g_{2n} - g_{2n+2}) + R_{2n-2, 2n}(g_{2n-2} - g_{2n+2}) = 0. \quad (4.7)$$

These three equations can be used together to obtain g_k for all k . For example, a knowledge of $R_{0,2}$ and $R_{1,3}$ (which we get from Eqs.(3.6) and (3.8)) will give γ_0 . This will give g_4 from (4.2) (again we need g_0 and g_2 which we will extract from Eq.(2.9)). A knowledge of g_4 in turn yields $R_{3,5}$ and $R_{2,4}$ (where we use $R_{0,0}$ and $R_{1,1}$, calculated from Eq.(3.7)) from Eqs.(4.6) and (4.7), which gives γ_1 . This in turn can be used to calculate g_6 (4.3) and so on.

However, we still need to know $R_{j,j}$ for $j \geq 2$. This can be obtained by writing $j = 2n+3$ and $k = 2n$ in Eq.(4.1). We get

$$R_{2n+3, 2n+3} R_{2n, 2n+2}(g_{2n+2} - g_{2n}) + R_{2n+3, 2n+5}(g_{2n} - g_{2n+4}) + R_{2n-1, 2n+1}(g_{2n} - g_{2n-2}) = 0. \quad (4.8)$$

$R_{2n+2, 2n+2}$ can be calculated from the relation $R_{2n+2, 2n+2} = -R_{2n+3, 2n+3}$. Thus a knowledge of g_4 , $R_{0,0}$ and $R_{0,2}$ gives $R_{3,5}$ (4.6). Then using g_4 , g_2 , $R_{3,5}$ and $R_{0,2}$, we can obtain $R_{3,3}$ (and hence $R_{2,2}$) from (4.8).

To summarize, we have outlined a recursive technique to obtain the normalization constant. This necessitates a knowledge of coefficients of the form $R_{k, k+2}$. Again to evaluate this (also needed in Eq.(3.17)), we need $R_{k,k}$. We obtain a series of self-consistent recursion relations to obtain these coefficients.

Comment: In the context of random matrix theory, we may point out that now we can obtain various statistical properties like the level-density and 2-point functions for even finite dimensional symplectic ensembles of random matrices ([1] [3] [4]).

5. THE ZEROS OF THESE POLYNOMIALS

Having obtained the polynomials for the specific case of $d = 2$, we will discuss briefly about the zeros of these SOP in this section. We know that in a given interval $[a, b]$

$$\forall n \neq 0, \quad \int_a^b \psi_n(x)w(x)dx = 0, \quad n \geq 2. \quad (5.1)$$

So $\psi_n(x)$ should have atleast one point in the interior of $[a, b]$ where it changes sign. Let there be k such points x_1, x_2, \dots, x_k . Then the function $\psi_n(x)(x - x_1) \dots (x - x_k)$ is positive or negative definite for $k \leq n + 2d - 1$, since $\psi_n(x)$ is a polynomial of order $n + 2d - 1$. This implies

$$\int_a^b \psi_n(x)(x - x_1) \dots (x - x_k)dx \neq 0, \quad \forall k \leq n + 2d - 1. \quad (5.2)$$

However, from skew-normalization relation, this condition is satisfied if and only if

1. For $n = 2m, k = 2m + 1$.
2. For $n = 2m + 1, k = 2m$.
3. $(x - x_1) \dots (x - x_k) = \phi_k(x)$.

This implies that

- a. $\phi_k(x)$ is a polynomial of order k with k real zeros.
- b. $\psi_k(x)$ is a polynomial of order $k + 2d - 1$, with $\psi_{2m}(x)$ having $2m + 1$ and $\psi_{2m+1}(x)$ having $2m$ real zeros.

6. CONCLUSION

This paper gives a formalism to derive the SOP arising in the study of symplectic ensembles of random matrices [6]. Here, we would like to point out that this paper contains almost no new results which were not present in [1] and [2]. However, a lot of the properties were overlooked. For example, the explicit form of the recursion coefficients, the necessity to use Gram-Schmidt method to obtain polynomials of order $n < 2d$, the zeros of $\phi_n(x)$ and $\psi_n(x)$ etc. At this point, one can easily obtain these polynomials, although a deeper insight into the recursion coefficients (specially its large n behavior) is an absolute necessity. However, with all its limitations, we hope that this paper will take us further towards our goal, which is “to develop the theory of skew orthogonal polynomials until it becomes a working tool as handy as the existing theory of orthogonal polynomials” [7].

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